



Note

Domination, radius, and minimum degree

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ARTICLE INFO

Article history:

Received 26 November 2008

Received in revised form 17 March 2009

Accepted 15 April 2009

Available online 9 May 2009

Keywords:

Domination

Radius

Minimum degree

ABSTRACT

We prove sharp bounds concerning domination number, radius, order and minimum degree of a graph. In particular, we prove that if G is a connected graph of order n , domination number γ and radius r , then $\frac{2}{3}r \leq \gamma \leq n - \frac{4}{3}r + \frac{2}{3}$. Equality is achieved in the upper bound if, and only if, G is a path or a cycle on n vertices with $n \equiv 4 \pmod{6}$. Further, if G has minimum degree $\delta \geq 3$ and $r \geq 6$, then using a result due to Erdős, Pach, Pollack, and Tuza [P. Erdős, J. Pach, R. Pollack, Z. Tuza, Radius, diameter, and minimum degree. J. Combin. Theory B 47 (1989), 73–79] we show that $\gamma \leq n - \frac{2}{3}(r - 6)\delta$.

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1. Introduction

In this paper, we continue the study of domination in graphs. Domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [5,6].

For notation and graph theory terminology we in general follow [5]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V . The *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. For a set S of vertices, the open neighborhood of S is defined by $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of S by $N[S] = N(S) \cup S$. If $X, Y \subseteq V$, then the set X is said to *dominate* the set Y if $Y \subseteq N[X]$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$ while the graph $G - S$ is the graph obtained from G by deleting the vertices in S and all edges incident with S . We denote the degree of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. The minimum degree among the vertices of G is denoted by $\delta(G)$, and the maximum degree by $\Delta(G)$. A cycle on n vertices is denoted by C_n , and a path on n vertices by P_n .

A *dominating set* of a graph $G = (V, E)$ is a set S of vertices of G such that every vertex $v \in V$ is either in S or adjacent to a vertex of S . (That is, $N[S] = V$.) The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$ -set.

A *total dominating set* of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S . (That is, $N(S) = V$.) The *total domination number* of G , denoted by $\gamma_t(G)$, is the minimum cardinality of a dominating set.

For two vertices u and v in a connected graph G , the *distance* $d_G(u, v)$ between u and v is the length of a shortest u – v path in G . A u – v path of length $d_G(u, v)$ is called a u – v *geodesic*. The *eccentricity* $e(v)$ of a vertex $v \in V$ is the distance between v and a vertex farthest from v in G . The minimum eccentricity among the vertices of G is its *radius* and the maximum eccentricity is its *diameter*, which are denoted by $\text{rad}(G)$ and $\text{diam}(G)$, respectively. A vertex v is a *central vertex* if $e(v) = \text{rad}(G)$ and the subgraph induced by the central vertices of G is the *center* $\text{Cen}(G)$ of G . A geodesic of length $\text{diam}(G)$ is called a *diametral path* in G . The concepts of radius and diameter are fundamental concepts in graph theory and are well-studied in the literature.

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2. Main results

Our aim in this paper is to establish relationships between the domination number and the radius of a graph in terms of its order, radius, and minimum degree.

We shall prove:

Theorem 1. *If G is a connected graph of order n , domination number γ , and radius r , then*

$$\frac{2}{3}r \leq \gamma \leq n - \frac{2}{3}(2r - 1).$$

The lower bound is sharp even for graphs with arbitrarily large, but fixed, minimum degree. Equality is achieved in the upper bound if and only if $G \in \{P_n, C_n\}$ and $n \equiv 4 \pmod{6}$.

The upper bound of [Theorem 1](#) can be improved if we restrict the minimum degree to be at least three and the radius to be at least six.

Theorem 2. *If G is a connected graph of order n , domination number γ , minimum degree $\delta \geq 3$ and radius $r \geq 6$, then*

$$\gamma \leq n - \frac{2}{3}(r - 6)\delta.$$

The bound is asymptotically best possible in the sense that there exist graphs satisfying the hypothesis of the theorem such that $\gamma = n - \frac{2}{3}(r + \frac{3}{2})\delta$.

3. Known results

In 1869, Jordan [8] showed that if T is a tree, then either $\text{diam}(T) = 2 \text{ rad}(T)$ and $\text{Cen}(T)$ contains exactly one vertex or $\text{diam}(T) = 2 \text{ rad}(T) - 1$ and $\text{Cen}(T)$ consists of two adjacent vertices. In particular, we have the following result.

Fact 1. *For every tree T , $\text{diam}(T) \geq 2 \text{ rad}(T) - 1$.*

The radius of a path or cycle is easy to compute.

Fact 2. *For $n \geq 1$, $\text{rad}(P_n) = \lfloor n/2 \rfloor$. For $n \geq 3$, $\text{rad}(C_n) = \text{rad}(P_n)$.*

We shall also need a useful result due to Erdős, Pach, Pollack, and Tuza [4] on radius, diameter, and minimum degree. To state their result, we introduce some notation.

Definition 1. Let $G = (V, E)$ be a connected graph with $\delta(G) \geq 3$ and $\text{rad}(G) = r \geq 6$. Let z be a fixed central vertex of G , and so $\text{rad}(G) = r = e(z)$. For each $i = 0, 1, \dots, r$, we define $V_i = \{v \in V \mid d_G(v, z) = i\}$. Hence, $V = (V_0, V_1, \dots, V_r)$ is a partition of V . We denote by $V_{\leq j}$ and $V_{\geq j}$ the sets $\bigcup_{0 \leq i \leq j} V_i$ and $\bigcup_{j \leq i \leq r} V_i$, respectively. Let T be a spanning tree of G that is distance-preserving from z ; that is, $d_T(v, z) = d_G(v, z)$ for all vertices $v \in V$. For a vertex $v \in V$, let $T(v, z)$ denote the set of vertices on the (unique) v - z path in T . Let $z_r \in V_r$. We say that a vertex $y \in V$ is *related* to the vertex z_r if there exist vertices $u, v \in V$, where $u \in T(z, z_r) \cap V_{\geq 5}$ and $v \in T(z, y) \cap V_{\geq 5}$ such that $d_G(u, v) \leq 2$.

We are now in a position to state the result due to Erdős et al.

Fact 3 (Erdős et al. [4]). *Let G be a connected graph with $\delta(G) \geq 3$ and $\text{rad}(G) = r \geq 6$ and let z be a central vertex of G . For each vertex $z_r \in V_r$, there exists a vertex in $V_{\geq r-5}$ which is not related to z_r .*

Let G be a connected graph. A vertex in G can be adjacent to at most three vertices of a shortest path in G . In particular, a vertex in a dominating set of G can dominate at most three vertices of a shortest path in G . Hence we have the following elementary result involving the diameter that gives a lower bound for the domination number.

Fact 4. *For every connected graph G , $\gamma(G) \geq \frac{1}{3}(\text{diam}(G) + 1)$.*

The domination of a path is easy to compute.

Fact 5. *For $n \geq 1$, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. For $n \geq 3$, $\gamma(C_n) = \gamma(P_n)$. Furthermore, if $n \equiv 1 \pmod{3}$ and v is any specified vertex of P_n , then there exists a $\gamma(P_n)$ -set that contains v .*

By [Fact 5](#), $\gamma(P_n) \leq (n + 2)/3$ with equality if and only if $n \equiv 1 \pmod{3}$. Since $\text{diam}(P_n) = n - 1$, we have the following result.

Fact 6. *For $n \geq 1$, $\gamma(P_n) \leq \frac{1}{3}\text{diam}(P_n) + 1$ with equality if and only if $n \equiv 1 \pmod{3}$.*

DeLaViña et al. [2] pointed out the following fundamental relationship between the total domination number and the radius of a graph.

Fact 7. For every connected graph G , $\gamma_t(G) \geq \text{rad}(G)$.

In 1996, Reed [9] presented the following important and useful result.

Fact 8 (Reed [9]). If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma(G) \leq 3n/8$.

In general, we have the following upper bound on the domination number of a graph.

Fact 9. If G is a graph with minimum degree $\delta \geq 1$ and order n , then $\gamma(G) \leq \left(\frac{1+\ln \delta}{\delta}\right)n$.

For δ large, the bound of Fact 9 is easily proven using probabilistic methods. It can be deduced from results of Alon [1] that the bound in Fact 9 is nearly optimal for large δ . A dominating set of cardinality $\frac{1}{\delta}(1 + \ln \delta)n$ can be constructed in complexity $O(n + \delta n)$ (see, for example, [7]).

4. The family \mathcal{G}_δ

In this section, for $\delta \geq 1$ we define a family of graphs \mathcal{G}_δ with minimum degree δ satisfying $\gamma(G) = \frac{2}{3}\text{rad}(G)$. For this purpose, let $k \geq 1$ be an integer and let $H = P_{6k}$ be a path of order $6k$. Let H be given by v_1, v_2, \dots, v_{6k} and let $S = \{v_i \mid i \equiv 2 \pmod{3}\}$. Note that S is the unique $\gamma(H)$ -set.

Let \mathcal{G}_1 be the family of all graphs obtained from H by attaching arbitrarily pendant vertices to some or all vertices of S . Then if $G \in \mathcal{G}_1$, we have $\delta(G) = 1$, $\text{rad}(G) = 3k$ and $\gamma(G) = |S| = 2k$. Thus, $\gamma(G) = \frac{2}{3}\text{rad}(G)$ for all graphs $G \in \mathcal{G}_1$.

For $\delta \geq 2$, let \mathcal{G}_δ be the family of all graphs G obtained from H by replacing the two end-vertices v_1 and v_{6k} by a complete graph K_δ , replacing the vertices v_i where $i \equiv 0 \pmod{3}$ and $i \neq 6k$ by a complete graph $K_{\lfloor \delta/2 \rfloor}$, replacing the vertices v_i where $i \equiv 1 \pmod{3}$ and $i \neq 1$ by a complete graph $K_{\lceil \delta/2 \rceil}$, and adding all edges between vertices from two cliques that correspond to adjacent vertices in H . Then, if $G \in \mathcal{G}_\delta$, we have $\delta(G) = \delta$, $|V(G)| = 2(\delta+1)k + \delta$, $\text{rad}(G) = 3k$ and $\gamma(G) = 2k$. In particular, $\gamma(G) = \frac{2}{3}\text{rad}(G)$ for all graphs $G \in \mathcal{G}_\delta$.

5. Proof of Theorem 1

Before we present a proof of Theorem 1, we shall need the following lemma. The proof is along similar lines to that presented in [2] for the total domination number.¹

Lemma 1. If S is a $\gamma(G)$ -set of a connected graph G , then there exists a spanning tree T of G such that S is a $\gamma(T)$ -set.

Proof. If G is a tree, then $T = G$ and result follows. So assume that G is not a tree and let C be a cycle in G . One of the following three cases must occur. Either C has two consecutive vertices a, b with $a \notin S$ and $b \notin S$ or C has two consecutive vertices a, b with $a \in S$ and $b \in S$ or C has three consecutive vertices a, b, c such that $a, c \in S$ and $b \notin S$. In all three cases, take $e = ab$. Then, e is edge on C such that S is a dominating set of the graph $G - e$. Repeating this process of deleting edges from cycles in G until there are no cycles remaining, produces a spanning tree T of G . By construction, the set S is a dominating set of T , and so $\gamma(T) \leq |S|$. Since adding edges to a graph cannot increase the domination number, we have that $|S| = \gamma(G) \leq \gamma(T)$. Consequently, $\gamma(T) = |S|$, and so S is a $\gamma(T)$ -set. \square

Recall the statement of the lower bound in Theorem 1.

Lower bound of Theorem 1. For every connected graph G , $\frac{2}{3}\text{rad}(G) \leq \gamma(G)$, and this bound is sharp even for graphs with arbitrarily large, but fixed, minimum degree.

Proof. Let S be a $\gamma(G)$ -set of a connected graph G . By Lemma 1, there exists a spanning tree T of G such that S is a $\gamma(T)$ -set. Hence, $\gamma(T) = \gamma(G)$. By Fact 4, we have that $\gamma(T) \geq \frac{1}{3}(\text{diam}(T) + 1)$. Thus, by Fact 1, $\gamma(G) \geq \frac{2}{3}\text{rad}(T)$. Since adding edges to a graph cannot increase the radius, $\text{rad}(T) \geq \text{rad}(G)$. Consequently, $\gamma(G) \geq \frac{2}{3}\text{rad}(G)$, as desired. That the lower bound of Theorem 1 is sharp, may be seen by considering the family \mathcal{G}_δ of graphs G constructed in Section 4 with minimum degree δ that satisfy $\gamma(G) = \frac{2}{3}\text{rad}(G)$. \square

Recall the statement of the upper bound in Theorem 1.

Upper bound of Theorem 1. If G is a connected graph of order n , then $\gamma(G) \leq n - \frac{4}{3}\text{rad}(G) + \frac{2}{3}$. Equality is achieved in the upper bound if and only if $G \in \{P_n, C_n\}$ and $n \equiv 4 \pmod{6}$.

¹ One of the referees of our manuscript kindly inform us that Lemma 1 and the lower bound of Theorem 1 have been independently proven by DeLaViña, Pepper, and Waller [3].

Proof. Let v be a vertex of G of maximum degree $\Delta(G)$ in G , and let T be a spanning tree of G that is distance-preserving from v . Then, $d_T(v) = d_G(v) = \Delta(G)$. Let P be a diametral path of T . Suppose that $T \neq P$. Let T_P be the forest obtained from T by deleting the vertices on the path P , i.e., $T_P = T - V(P)$. Since adding edges to a graph cannot increase the domination number,

$$\gamma(G) \leq \gamma(T) \leq \gamma(P) + \gamma(T_P). \quad (1)$$

Let $d = \text{diam}(T)$. Since P is a path on $d + 1$ vertices, we have $\gamma(P) = \gamma(P_{d+1})$, and so by Fact 6,

$$\gamma(P) \leq \frac{1}{3}d + 1, \quad (2)$$

with equality if and only if $d + 1 \equiv 1 \pmod{3}$. Further,

$$\gamma(T_P) \leq n - d - 1, \quad (3)$$

with equality if and only if T_P is the empty graph \bar{K}_{n-d-1} (of order $|V(T)| - |V(P)| = n - d - 1$ and size 0). By Eqs. (1)–(3), we have that

$$\gamma(G) \leq n - \frac{2}{3}d. \quad (4)$$

Hence by Fact 1, $\gamma(G) \leq n - \frac{2}{3}(2 \text{ rad}(T) - 1)$. Thus since $\text{rad}(T) \geq \text{rad}(G)$, we have that $\gamma(G) \leq n - \frac{4}{3} \text{ rad}(G) + \frac{2}{3}$, which is the desired upper bound. However suppose that in this case (when $T \neq P$) there exists a graph G achieving equality in the upper bound of Theorem 1. Then all the above inequalities must be equalities. In particular, by Eq. (2), $P = P_{d+1}$ where $d + 1 \equiv 1 \pmod{3}$ and, by Eq. (3), $T_P = \bar{K}_{n-d-1}$. Thus, $V(T) \setminus V(P)$ is an independent set in T . Let $w \in V(T_P)$ and let v be the (unique) neighbor of w in T . Necessarily, $v \in V(P)$. By Fact 5, there exists a $\gamma(P)$ -set S_v that contains v . The set $S_v \cup (V(T_P) \setminus \{w\})$ is a dominating set of T of cardinality $|S_v| + |V(T_P)| - 1 = \gamma(P) + \gamma(T_P) - 1$, and so $\gamma(T) < \gamma(P) + \gamma(T_P)$, contradicting the fact that we have equality in Eq. (1). Hence in this case, there is no graph G achieving equality in the upper bound of Theorem 1. That is, if $T \neq P$, then $\gamma(G) < n - \frac{4}{3} \text{ rad}(G) + \frac{2}{3}$.

Suppose, then, that $T = P$. Then, $\Delta(G) \leq 2$, and so $G \in \{P_n, C_n\}$. By Facts 2 and 5, we have that

$$n - \frac{4}{3} \text{ rad}(G) + \frac{2}{3} = n - \frac{4}{3} \cdot \left\lfloor \frac{n}{2} \right\rfloor + \frac{2}{3} \geq n - \frac{4}{3} \cdot \frac{n}{2} + \frac{2}{3} = \frac{n+2}{3} \geq \left\lceil \frac{n}{3} \right\rceil = \gamma(G), \quad (5)$$

and the desired upper bound holds. Furthermore, if we have equality in the upper bound of Theorem 1 (and still $G \in \{P_n, C_n\}$), then we must have equality throughout the Inequality Chain (5). In particular, $\lfloor n/2 \rfloor = n/2$ and $(n+2)/3 = \lceil n/3 \rceil$. This implies that n is even and $n \equiv 1 \pmod{3}$. Consequently, $n \equiv 4 \pmod{6}$. \square

6. Proof of Theorem 2

Recall the statement of Theorem 2.

Theorem 2. If G is a connected graph of order n , minimum degree $\delta \geq 3$ and radius $r \geq 6$, then

$$\gamma(G) \leq n - \frac{2}{3}(r - 6)\delta.$$

The bound is asymptotically best possible in the sense that there exist graphs satisfying the hypothesis of the theorem such that $\gamma(G) = n - \frac{2}{3}(r + \frac{3}{2})\delta$.

Proof. Let z be a fixed central vertex of G . Using the notation defined in Definition 1, by Fact 3, there is a fixed vertex $x \in V_{r-5}$ which is not related to $z_r \in V_r$. Let $T(z, z_r)$ be the path $z = y_0, y_1, y_2, \dots, y_{r-1}, y_r = z_r$. Thus, $y_i \in V_i$ for $i = 0, 1, \dots, r$. Set $P = \{y_{3i+1} \mid i = 0, 1, \dots, q-1\}$, where $q = \lfloor \frac{r+1}{3} \rfloor$. Since $T(z, z_r)$ is a shortest path in G , we have

$$N[u] \cap N[v] = \emptyset \quad \text{for all } u, v \in P. \quad (6)$$

Let $T(z, x)$ be the path $z = x_0, x_1, x_2, \dots, x_{r-6}, x_{r-5} = x$. Thus, $x_i \in V_i$ for $i = 0, 1, \dots, r-5$. Set $Q = \{x_{3i+1} \mid i = 2, 3, \dots, t+1\}$, where $t = \lfloor \frac{r-10}{3} \rfloor$. Since $T(z, x)$ is a shortest path in G , we have

$$N[u] \cap N[v] = \emptyset \quad \text{for all } u, v \in Q. \quad (7)$$

We now define the set A to be the set $A = P \cup Q$ and we define $B = V - N[A]$. Since x and z_r are unrelated, Properties (6) and (7) imply that

$$N[u] \cap N[v] = \emptyset \quad \text{for all } u, v \in A. \quad (8)$$

Since G has minimum degree δ , Property (8) implies that

$$n = |N[A]| + |B| \geq |A|(\delta + 1) + |B|,$$

or, equivalently,

$$|B| \leq n - |A|(\delta + 1). \quad (9)$$

Since $A \cup B$ is a dominating set of G , we have by Eq. (9) that

$$\begin{aligned} \gamma(G) &\leq |A| + |B| \\ &\leq |A| + n - |A|(\delta + 1) \\ &= n - \delta|A| \\ &= n - \delta \left(\left\lfloor \frac{r+1}{3} \right\rfloor + \left\lfloor \frac{r-10}{3} \right\rfloor \right) \\ &\leq n - \frac{2}{3}(r-6)\delta, \end{aligned}$$

as desired. That the upper bound of Theorem 2 is asymptotically best possible, may be seen by considering the family \mathcal{G}_δ , $\delta \geq 3$, of graphs G constructed in Section 4 with minimum degree δ and radius $r \geq 6$ that satisfy $\gamma(G) = n - \frac{2}{3}(r + \frac{3}{2})\delta$. \square

Remark: Let G be a connected graph of order n , domination number γ , minimum degree $\delta \geq 3$ and radius $r \geq 6$. Note that if $\delta = 3$ and $r > \frac{5}{16}n + 6$, then the upper bound in Theorem 2 for γ is an improvement to Reed's upper bound of $3n/8$ in Fact 8. If $r > \frac{3}{2} \frac{1}{\delta^2}(\delta - \ln \delta - 1)n + 6$, then the upper bound in Theorem 2 for γ is an improvement to the bound $(\frac{1+\ln \delta}{\delta})n$ presented in Fact 9. Although the existence of such graphs is established in Section 4 and in [4] (Theorem 1), we do not have any guess as to how many such graphs there are.

Acknowledgements

First author's research was supported in part by the South African National Research Foundation and the University of KwaZulu-Natal. Financial support by the Competitive Research Grant from the University of KwaZulu-Natal is gratefully acknowledged.

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